An integrable system connected with the Uvarov-Chihara problem for orthogonal polynomials

1998 J. Phys. A: Math. Gen. 31 9579

(http://iopscience.iop.org/0305-4470/31/47/017)

J. Phys. A: Math. Gen. **31** (1998) 9579–9591. Printed in the UK PII: S0305-4470(98)91494-9

An integrable system connected with the Uvarov–Chihara problem for orthogonal polynomials

Luc Vinet† and Alexei Zhedanov‡

† Centre de Recherches Mathématiques, Université de Montréal, CP 6128, succursale Centre-ville, Montréal, Québec, Canada, H3C 3J7

‡ Donetsk Institute for Physics and Technology, Donetsk 340114, Ukraine

Received 10 February 1998, in final form 10 September 1998

Abstract. A new non-local integrable chain with continuous time and discrete space variable is considered. In contrast to the case of Toda and Volterra chains, the general solution can be presented in explicit form in terms of two arbitrary sequences. It is shown that this solution is connected with the so-called Uvarov–Chihara problem (inserting a discrete mass at the centre of the spectral interval of symmetric orthogonal polynomials). The asymptotic behaviour of the recurrence coefficients of such polynomials is considered.

1. Introduction

Consider a system $P_n(x)$ of symmetric orthogonal polynomials satisfying the recurrence relation

$$P_{n+1} + u_n P_{n-1} = x P_n \qquad n = 1, 2, \ldots \tag{1.1}$$

with the initial conditions

$$P_0 = 1 \qquad P_1 = x. \tag{1.2}$$

Then equation (1.1), together with (1.2), defines a system of monic polynomials

$$P_n(x) = x^n + \mathrm{O}(x^{n-2}). \tag{1.3}$$

If all $u_n > 0$, there exists an even weight function $w(x)$ such that (see, e.g., [1, 2])

$$\int_{-a}^{a} P_n(x) P_m(x) w(x)\, \mathrm{d}x = h_n \delta_{nm} \tag{1.4}$$

where the integral is understood to be of Stieltjes type (i.e. the measure $\mathrm{d}\mu(x) = w(x)\,\mathrm{d}x$ may contain continuous, discrete and singular continuous parts; in two last cases $w(x)$ involves Dirac δ-functions). The spectral interval may be either finite or infinite: $a \leqslant \infty$. The normalization constants are $h_0 = 1,\ h_n = u_1 u_2 \cdots u_n$.

Assume that the recurrence coefficients $u_n(t)$ are functions of the time t—an additional parameter. Thus the polynomials P_n will also be functions of t. We shall indicate this by using the notation $P_n(x;t)$ (as usual, $\dot{f}$ will stand for the time derivative of a function f). Then obviously we have the following expansion:

$$\dot{P}_n(x;t) = C^{(n)}_{n-2}(t) P_{n-2}(x;t) + \cdots + C^{(n)}_{n-2k}(t) P_{n-2k}(x;t) + \cdots \tag{1.5}$$

where the last term is proportional to $\sim P_0$ for n even and $\sim P_1$ for n odd. Clearly, the coefficients $C^{(n)}_k(t)$ in the expansion (1.5) define the time dynamics of $u_n(t)$ and $P_n(x;t)$.

It is of interest to consider the integrable cases of such systems and thus to determine the coefficients $C_k^{(n)}(t)$ for which the recurrence relation (1.1) will be valid for all values of t.

Such a problem is well known in the mathematical physics literature and is referred to as the 'Volterra hierarchy' problem (see, e.g. [3]). For example, if we restrict ourselves to the simplest case

$$\dot{P}_n(x,t) = C_{n-2}(t)P_{n-2}(x;t) \tag{1.6}$$

the compatibility condition of (1.1) and (1.6) yields

$$C_{n-2}(t) = -u_n u_{n-1} \tag{1.7}$$

(up to an arbitrary constant factor) leading to the well known Volterra chain

$$\dot{u}_n = u_n(u_{n+1} - u_{n-1}). \tag{1.8}$$

The weight function of the corresponding polynomials is

$$w(x;t) = \kappa(t)w(x;0)\exp(x^2 t) \tag{1.9}$$

where the normalization factor $\kappa(t)$ is such that the condition $h_0(t) = h_0(0) = 1$ is satisfied.

If in addition one allows the term proportional to P_{n-4} in (1.5), then one obtains a more complicated equation for u_n containing the terms $u_{n-2}, u_{n-1}, \ldots, u_{n+2}$. In general, keeping terms up to order $n-2k$ in (1.5), one obtains an equation for u_n containing the terms $u_{n-k}, \ldots, u_{n+k}$. What happens, then, if all terms in (1.5) are retained? Obviously, one then obtains a *non-local* equation for u_n. Here we present one such a system with very simple properties; it turns out to be related to the Uvarov problem [4], which is concerned with how to add a discrete mass to an initial weight function of the orthogonal polynomials.

2. The integrable system and its solutions

We introduce the following ansatz for the time dynamics of P_n:

$$\dot{P}_{2n+1} = 0 \qquad \dot{P}_{2n} = r_n x^{-1} P_{2n-1} \tag{2.1}$$

where the r_n are some coefficients depending on t. It can easily be seen that equation (2.1) is a special case of the Volterra hierarchy (1.5). Indeed, using the Christoffel–Darboux formula [1, 2]

$$\frac{P_n(x)P_{n-1}(y) - P_n(y)P_{n-1}(x)}{(x-y)h_{n-1}} = \sum_{k=0}^{n-1} \frac{P_k(x)P_k(y)}{h_k} \tag{2.2}$$

setting $y = 0$ and taking into account the fact that $P_{2n+1}(0) = 0$ for symmetric polynomials, we can rewrite (2.1) in the form

$$\dot{P}_{2n}(x) = \sum_{k=0}^{n-1} C_{2k}^{(n)} P_{2k}(x) \tag{2.3}$$

where

$$C_{2k}^{(n)} = \frac{r_n h_{2n-2} P_{2k}(0)}{h_{2k} P_{2n-2}(0)} \tag{2.4}$$

where

$$P_{2n}(0) = (-1)^n u_1 u_3 \cdots u_{2n-1}. \tag{2.5}$$

Since all $C_{2k}^{(n)}$ are generally non-zero, we deal with non-local equations for u_n. These equations are straightforwardly derived from (2.1) and (1.1):

$$\dot{u}_{2n} = r_n \qquad \dot{u}_{2n+1} = -r_{n+1} \tag{2.6}$$

where the coefficients $r_n(t)$ satisfy the relations

$$\frac{r_{n+1}}{r_n} = \frac{u_{2n+1}}{u_{2n}}. \tag{2.7}$$

From equations (2.7) we find the explicit expression for r_n:

$$r_n = r_1 \frac{u_3 u_5 \cdots u_{2n-1}}{u_2 u_4 \cdots u_{2n-2}} \qquad n = 2, 3, \ldots \tag{2.8}$$

where $r_1(t)$ is an arbitrary function of time. Without loss of generality we can choose

$$r_1 = 1. \tag{2.9}$$

(Making other choices for $r_1(t)$ merely correspond to changing of the variable t.)

In spite of its non-locality, the system (2.6) looks much simpler than the Volterra equation (1.8). Moreover, its general solution can be constructed explicitly.

Indeed, consider the first four equations of the system (2.6) (taking into account the condition (2.9))

$$\dot{u}_1 = -1 \qquad \dot{u}_2 = 1 \qquad \dot{u}_3 = -\frac{u_3}{u_2} \qquad u_4 = \frac{u_3}{u_2}. \tag{2.10}$$

Their solutions are

$$\begin{aligned} &u_1(t) = u_1(t_0) - t + t_0 \qquad u_2(t) = u_2(t_0) + t - t_0 \\ &u_3(t) = \frac{u_2(t_0)u_3(t_0)}{u_2(t_0) + t - t_0} \qquad u_4(t) = u_4(t_0) + u_3(t_0) - \frac{u_2(t_0)u_3(t_0)}{u_2(t_0) + t - t_0} \end{aligned} \tag{2.11}$$

where $u_i(t_0)$ are the values of the recurrence coefficients at t_0. It is obvious that all other coefficients $u_n(t)$ can be found in a similar (step-by-step) manner from (2.6).

There is, however, a more systematic way to find the general solution of the system (2.6). Indeed, note that the system (2.6) admits the two integrals of motion

$$D_n = u_{2n}(t)u_{2n+1}(t) \qquad E_n = -u_{2n+1}(t) - u_{2n+2}(t) \tag{2.12}$$

where D_n, E_n do not depend on t.

From equations (2.12) we find the discrete Riccati equation for u_{2n}:

$$\frac{D_n}{u_{2n}} + u_{2n+2} = -E_n. \tag{2.13}$$

This equation is linearized by the substitution

$$u_{2n} = \frac{\varphi_n}{\varphi_{n-1}} \tag{2.14}$$

and we obtain a discrete Schrödinger equation (without a spectral parameter) for φ_n:

$$\varphi_{n+1} + E_n\varphi_n + D_n\varphi_{n-1} = 0. \tag{2.15}$$

Since D_n and E_n do not depend on time, we can choose two linearly independent solutions ξ_n and η_n which are independent of t. The general solution can then be written as

$$\varphi_n(t) = A_1(t)\xi_n + A_2(t)\eta_n \tag{2.16}$$

where $A_{1,2}(t)$ are two arbitrary functions. Hence we obtain

$$u_{2n}(t) = \frac{\xi_n + f(t)\eta_n}{\xi_{n-1} + f(t)\eta_{n-1}} \tag{2.17}$$

where $f(t) = A_2(t)/A_1(t)$.

From equations (2.12) we find

$$u_{2n+1}(t) = D_n \frac{\xi_{n-1} + f(t)\eta_{n-1}}{\xi_n + f(t)\eta_n}. \tag{2.18}$$

Equations (2.17) and (2.18) yield the general solution of (2.6), and moreover for r_n we obtain the expression

$$r_n = \dot{u}_{2n} = \frac{\dot{f}(t)W_n}{(\xi_{n-1} + f(t)\eta_{n-1})^2} \tag{2.19}$$

where

$$W_n = \eta_n\xi_{n-1} - \eta_{n-1}\xi_n \tag{2.20}$$

is the discrete Wronskian obeying the relation

$$\frac{W_{n+1}}{W_n} = D_n. \tag{2.21}$$

The function $f(t)$ is determined from (2.19) using the condition $r_1 = 1$:

$$f(t) = \eta_0^{-1}\left(\frac{\omega}{t-c} - \xi_0\right) \tag{2.22}$$

where

$$\omega = \xi_1 - \frac{\eta_1\xi_0}{\eta_0} \tag{2.23}$$

and c is an arbitrary constant (defining the value $f(t_0)$).

We thus arrive at the following result.

Proposition 1. Let ξ_n and η_n be two arbitrary sequences subject to the condition that

$$W_n \neq 0 \qquad \text{for all } n. \tag{2.24}$$

Equations (2.17), (2.18) yield the general solution of the system (2.1) for $r_1 = 1$, and with D_n and $f(t)$ defined in (2.21) and (2.22).

Note that the complete integrability of the system (2.1) is an obvious consequence of the fact that the spectral parameter x does not depend on t. Hence there are infinitely many integrals of motion of the type $I_k = \mathrm{Tr}(L^k)$ where the difference operator L is defined by

$$L|0\rangle = u_1|1\rangle \qquad L|n\rangle = u_{n+1}|n+1\rangle + |n-1\rangle \tag{2.25}$$

for some basis $|n\rangle$, $n = 0, 1, \ldots$.

3. The time dependence of the weight function

In this section we find the time dependence of the weight function $w(x;t)$ of the polynomials $P_n(x;t)$.

Proposition 2. Assume that $w(x;t_0) \neq 0$. Then the weight function has the following dependence on time:

$$w(x;t) = (1 - J(t))w(x;t_0) + J(t)\delta(x) \tag{3.1}$$

where the function $J(t)$ is

$$J(t) = 1 - \exp\left(-\int_{t_0}^{t} \frac{\mathrm{d}\tau}{u_1(\tau)}\right). \tag{3.2}$$

Proof. From the orthogonality relation we have

$$\int_{-a}^{a} P_{2n+2}(x;t)\dot{w}(x;t)\,\mathrm{d}x = -r_{n+1}(t)\int_{-a}^{a} x^{-1}P_{2n+1}w\,\mathrm{d}x. \tag{3.3}$$

On the other hand, from the Christoffel–Darboux formula (2.2) we have

$$\frac{P_{2n+1}(x)}{x} = \frac{h_{2n}}{P_{2n}(0)}\sum_{k=0}^{n}\frac{P_{2k}(x)P_{2k}(0)}{h_{2k}}. \tag{3.4}$$

Taking into account equation (2.5), from equations (3.3) and (3.4) for *any* even polynomial we obtain the identity

$$\int_{-a}^{a} P_{2n+2}(x;t)\dot{w}(x;t)\,\mathrm{d}x = \frac{P_{2n+2}(0;t)}{u_1(t)}. \tag{3.5}$$

Moreover

$$\int_{-a}^{a}\dot{w}(x;t)\,\mathrm{d}x = 0 \tag{3.6}$$

since $w(x;t)$ is normalized. Hence from (3.5) and (3.6) we arrive at the formula

$$\dot{w}(x;t) = \frac{\delta(x) - w(x;t)}{u_1(t)}. \tag{3.7}$$

We introduce the functions

$$F(t) = \exp\left(-\int_{t_0}^{t}\frac{\mathrm{d}\tau}{u_1(\tau)}\right)$$

$$H(t) = \int_{t_0}^{t}\frac{\mathrm{d}\tau}{F(\tau)u_1(\tau)}.$$

Then the general solution of equation (3.7) is written as

$$w(x;t) = F(t)\left(H(t)\delta(x) + w(x;t_0)\right). \tag{3.8}$$

Using the explicit expression (2.11) for $u_1(t)$ we obtain

$$F(t) = \frac{u_1(t_0) + t_0 - t}{u_1(t_0)} \qquad H(t) = \frac{t - t_0}{u_1(t_0) + t_0 - t}. \tag{3.9}$$

We then arrive at equation (3.1), where

$$J(t) = 1 - F(t) = \frac{t - t_0}{u_1(t_0)}. \tag{3.10}$$

Thus we have proved the proposition. As a byproduct we have obtained the result that $J(t)$ is a linear function in the time t, equation (3.10). Note also that the normalization condition

$$\int_{-a}^{a} w(x;t)\,\mathrm{d}x = 1 \tag{3.11}$$

is fulfilled automatically for all values of t, as can be seen from (3.1).

The converse statement is also valid. □

Proposition 3. Let $P_n(x;t)$ be a system of symmetric orthogonal polynomials whose weight function is defined by (3.1) with the function $J(t)$ defined by (3.2), then equations (2.1) are valid.

We will not prove this statement, because it is a direct consequence of the so-called Uvarov–Chihara transformation for orthogonal polynomials to be considered in section 4.

4. Connection with the Uvarov–Chihara problem

The system (2.1) is closely related to the so-called Uvarov–Chihara problem [4, 5] of how to insert a discrete mass M in a given weight function of the orthogonal polynomials $P_n(x)$. In our case we deal with the special case (considered in detail by Chihara [5]): how to insert a mass M at the origin (i.e. the centre of the spectral interval) for symmetric orthogonal polynomials. In this section we recall some results of [4, 5] using Uvarov's method [4].

Assume that the symmetric monic polynomials $P_n(x)$ have (the normalized) weight function $w(x)$, whereas the polynomials $\tilde{P}_n(x)$ have the weight function

$$\tilde{w}(x) = (1 - J)w(x) + J\delta(x) \tag{4.1}$$

with J some constant. Obviously, the weight function $\tilde{w}(x)$ is also normalized $\int_{-a}^{a} \tilde{w}(x)\,\mathrm{d}x = 1$. Note also that the weight function $\tilde{w}(x)/(1 - J)$ differs from $w(x)$ by the mass $M = J/(1 - J)$ inserted at the centre $x = 0$ of the spectral interval.

We can present the polynomials $\tilde{P}_n(x)$ as a superposition of the polynomials $P_n(x)$:

$$\tilde{P}_n(x) = P_n(x) + \sum_{k=0}^{n-1} C_k^{(n)} P_k(x) \tag{4.2}$$

where the $C_k^{(n)}$ are some coefficients to be determined. From the orthogonality relation we have

$$\begin{aligned} C_k^{(n)} h_k &= \int_{-a}^{a} \tilde{P}_n(x) P_k(x) w(x)\,\mathrm{d}x \\ &= \int_{-a}^{a} \tilde{P}_n(x) P_k(x) \left(\frac{\tilde{w}(x) - J\delta(x)}{1 - J} \right) \mathrm{d}x = \frac{J}{J - 1} \tilde{P}_n(0) P_k(0) \end{aligned} \tag{4.3}$$

(the last equality is due to orthogonality of the polynomials $\tilde{P}_n(x)$ with respect to *any* polynomial of a lesser degree (say, $P_k(x)$)). For symmetric orthogonal polynomials $\tilde{P}_{2m+1}(0) = P_{2k+1}(0) = 0$. Hence we have

$$C_{2m+1}^{(n)} = C_k^{(2m+1)} = 0. \tag{4.4}$$

This means in particular that

$$\tilde{P}_{2n+1}(x) = P_{2n+1}(x) \tag{4.5}$$

i.e. that the polynomials with odd degree remain unchanged under the Uvarov–Chihara transformation. For the polynomials with even degree we have the following expansion from (4.3):

$$\tilde{P}_{2n}(x) = P_{2n}(x) + M\tilde{P}_{2n}(0) \sum_{k=0}^{n-1} \frac{P_{2k}(0) P_{2k}(x)}{h_{2k}}. \tag{4.6}$$

Using the Christoffel–Darboux formula (2.2) we can rewrite the right-hand side of (4.6) in the form

$$\tilde{P}_{2n}(x) = P_{2n}(x) - M\tilde{P}_{2n}(0) \frac{P_{2n}(0) P_{2n-1}(x)}{x h_{2n-1}} = \frac{P_{2n+1}(x) + \tilde{u}_{2n} P_{2n-1}(x)}{x} \tag{4.7}$$

where we introduced the new recurrence coefficients

$$\tilde{u}_{2n} = u_{2n} - M \frac{\tilde{P}_{2n}(0) P_{2n}(0)}{h_{2n-1}}. \tag{4.8}$$

Relations (4.7) and (4.8) follow from the recurrence relation

$$x\tilde{P}_{2n}(x) = \tilde{P}_{2n+1}(x) + \tilde{u}_{2n}\tilde{P}_{2n-1}(x) = P_{2n+1}(x) + \tilde{u}_{2n}P_{2n-1}(x). \tag{4.9}$$

In order to find $\tilde{P}_{2n}(0)$ we set $x = 0$ in (4.7):

$$\tilde{P}_{2n}(0) = \frac{P_{2n}(0)}{1 + M(P_{2n}(0)P'_{2n-1}(0)/h_{2n-1})}. \tag{4.10}$$

Thus for the recurrence coefficients we obtain

$$\tilde{u}_{2n} = u_{2n} - \frac{P^2_{2n}(0)}{P_{2n}(0)P'_{2n-1}(0) + h_{2n-1}/M}. \tag{4.11}$$

From the recurrence relations for $P_n(x)$ and $\tilde{P}_n(x)$ we easily find that

$$\tilde{u}_{2n}\tilde{u}_{2n+1} = u_{2n}u_{2n+1} \qquad \tilde{u}_{2n-1} + \tilde{u}_{2n} = u_{2n-1} + u_{2n}. \tag{4.12}$$

From equation (4.11) and the last relation of (4.12) we obtain the following expression for the recurrence coefficients with odd indices:

$$\tilde{u}_{2n-1} = u_{2n-1} + \frac{P^2_{2n}(0)}{P_{2n}(0)P'_{2n-1}(0) + h_{2n-1}/M}. \tag{4.13}$$

Equations (4.11), (4.13) and (4.7) thus yield the complete solution of the Uvarov–Chihara problem. Note that if $M \to 0$, then $\tilde{u}_n \to u_n$ as expected.

Note that for $P'_{2n+1}(0)$ we have the recurrence relation

$$P'_{2n+1}(0) + u_{2n}P'_{2n-1}(0) = P_{2n}(0) = (-1)^n u_1 u_3 \cdots u_{2n-1} \tag{4.14}$$

whence one obtains the following expression:

$$P'_{2n+1}(0) = (-1)^n u_2 u_4 \cdots u_{2n}\left(1 + \sum_{k=1}^{n} \frac{u_1 u_3 \cdots u_{2k-1}}{u_2 u_4 \cdots u_{2k}}\right). \tag{4.15}$$

Returning to our integrable system, we can introduce the time variable t by means of the relation (see equation (3.10))

$$J(t) = \frac{t - t_0}{u_1(t_0)} \tag{4.16}$$

in such a manner that $J(t_0) = 0$, i.e. for $t = t_0$ we have the 'unperturbed' polynomials $P_n(x)$, whereas for other values of t we have $\tilde{P}_n(x)$. This leads to proposition 3.

It is useful to rewrite the formulae providing the solution of the Uvarov–Chihara problem in a time evolution form:

$$\begin{aligned} P_{2n+1}(x;t) &= P_{2n+1}(x;t_0) \\ P_{2n}(x;t) &= \frac{P_{2n+1}(x;t_0) + u_{2n}(t)P_{2n-1}(x;t_0)}{x} \end{aligned} \tag{4.17}$$

where the time dependence of the recurrence coefficients is given by the formulae

$$u_{2n}(t) = u_{2n}(t_0) - \frac{P^2_{2n}(0;t_0)}{P_{2n}(0;t_0)P'_{2n-1}(0;t_0) + h_{2n-1}(t_0)/M(t)} \tag{4.18}$$

$$u_{2n-1}(t) = u_{2n-1}(t_0) + \frac{P^2_{2n}(0;t_0)}{P_{2n}(0;t_0)P'_{2n-1}(0;t_0) + h_{2n-1}(t_0)/M(t)}. \tag{4.19}$$

Note that the time dependence on the right-hand sides of (4.18) and (4.19) is contained only in $M(t)$:

$$M(t) = \frac{t - t_0}{u_1(t_0) - t + t_0}. \tag{4.20}$$

We can now compare equations (4.18), (4.19) with equations (2.17) and (2.18) which present the same solution in slightly different forms. It can easily be found that the sequences ξ_n and η_n must be

$$\xi_{n-1} = (-1)^n P'_{2n-1}(0; t_0) \tag{4.21}$$

$$\eta_{n-1} = (-1)^n \frac{h_{2n-1}(t_0)}{P_{2n}(0; t_0)} = u_2 u_4 \cdots u_{2n} \qquad n \geqslant 2 \tag{4.22}$$

with initial values $\eta_0 = -\xi_0 = 1$. As for the function $f(t)$ in (2.17), (2.18), we have the expression

$$f(t) = \frac{1}{M(t)} = -\frac{t - t_0 - u_1(t_0)}{t - t_0}. \tag{4.23}$$

Thus we have related the sequences ξ_n, η_n and the function $f(t)$ (giving the arbitrary solution of our integrable system) to the parameters of the corresponding orthogonal polynomials $P_n(x; t_0)$.

5. Asymptotics of the recurrence coefficients

It is well known that there are some necessary conditions for a Schrödinger potential $u(x)$ to possess a discrete level inside a continuous spectrum (see, e.g., [6]:

(i) $u(x)$ should be oscillating in x;
(ii) $u(x)$ should tend to zero as $\sim 1/x$.

We shall show that similar conditions are necessary for the symmetric orthogonal polynomials to have a discrete level at the centre of the spectral interval.

Indeed, let us assume that the 'initial' polynomials (for $t = t_0$) have the following asymptotics for the recurrence coefficients

$$u_n - 1 = \mathrm{O}\left(\frac{1}{n^3}\right). \tag{5.1}$$

Condition (5.1) guarantees that the spectral interval is $[-2, 2]$ (apart from a possible finite number of discrete masses outside this interval)—see, e.g., [7]. Then from equations (2.17), (2.18), we find that asymptotically ($n \to \infty$)

$$\eta_n \sim \text{constant} \qquad W_n \sim \text{constant} \qquad \xi_n \sim \alpha n + \beta \tag{5.2}$$

where α, β are some constants. Hence from equations (2.17), (2.18), for the coefficients we have

$$u_{2n} \sim 1 + \frac{1}{n} \qquad u_{2n+1} \sim 1 - \frac{1}{n}. \tag{5.3}$$

We see that conditions (i) and (ii) for the discrete potentials can be expressed in the form:

(i) the potentials u_n oscillate about 1;
(ii) $|u_n - 1| \sim 1/n$.

It is interesting to note that the asymptotic behaviour (5.3) *does not depend* on the value of the mass J. This dependence only appears in the next approximation $O(1/n^2)$. Hence the insertion of any small discrete mass M into the centre of spectral interval leads to drastic change in the asymptotic behaviour of the recurrence coefficients ($|u_n - 1| \sim 1/n$ instead of $|u_n - 1| \sim 1/n^3$ as expected for sufficiently 'good' potentials u_n), giving rise to a purely continuous spectrum on the interval $[-2, 2]$.

It is not clear whether the conditions (5.3) are *sufficient* for the existence of a discrete mass inserted at the centre of the spectral interval.

6. Some explicit examples

Equations (2.17), (2.18) allow one to construct many explicit examples starting from two arbitrary sequences ξ_n and η_n. However, if we wish to start from polynomials having strictly $[-2, 2]$ as the spectral interval, one should then impose restrictions such as (5.2) to these sequences.

The first possibility is to take the conditions (5.2) as *exact* for all $n = 0, 1, \ldots$. Namely to set

$$\xi_n = \alpha n + \beta \qquad \eta_n = \eta \tag{6.1}$$

with η, α, β some constants. From the initial conditions $\xi_0 = -1$, $\eta_0 = 1$ we find that $\eta = 1$, $\beta = -1$. Moreover, from relations (4.21) and (4.22) we easily find that the recurrence parameter $u_1(t_0)$ can be chosen arbitrarily, then $\alpha = -u_1(t_0)$ and all subsequent recurrence parameters are identical

$$u_2(t_0) = u_3(t_0) = \cdots = 1. \tag{6.2}$$

The orthogonal polynomials $P_n(x; t_0)$ corresponding to the recurrence parameters (6.2) (with arbitrary u_1) were considered by Geronimus [8] who showed that for $u_1 > 2$ the weight function for these polynomials can be written in the form

$$w(x; t_0) = \frac{\mathrm{e}^{-\omega} \cosh \omega}{\pi} \left(\frac{\sqrt{4 - x^2}}{4 \cosh^2 \omega - x^2} + \pi \tanh \omega \big(\delta(x - x_0) + \delta(x + x_0) \big) \right) \tag{6.3}$$

where $\mathrm{e}^{2\omega} = u_1 - 1$ and $x_0 = 2 \cosh \omega$. This means that the weight function (6.3) contains a continuous part on the interval $[-2, 2]$ and two additional masses located at the points $\pm x_0$ beyond this interval.

The polynomials $P_n(x; t_0)$ have a simple explicit expression in terms of the Chebyshev polynomials $U_n(x)$ of the second kind [8]:

$$P_n(x; t_0) = U_n(x) - (u_1(t_0) - 1) U_{n-2}(x) \tag{6.4}$$

where the monic Chebyshev polynomials are defined as

$$U_n(x) = \frac{\sin(\theta(n+1))}{\sin \theta} \qquad x = 2 \cos \theta. \tag{6.5}$$

(In order for equation (6.4) to be valid for $n = 0, 1$ we should define $U_{-k}(x) = 0$). It is clear that the case $u_1(t_0) = 1$ corresponds to the Chebyshev polynomials $U_n(x)$ whereas the case $u_1(t_0) = 2$ corresponds to the Chebyshev polynomials $T_n(x)$ of the first kind.

Upon performing the Uvarov–Chihara transformation we can introduce the time variable t in such a way that the new recurrence coefficients will be

$$\begin{aligned} u_{2n}(t) &= \frac{(t - t_0)n + 1}{(t - t_0)(n - 1) + 1} \\ u_{2n+1}(t) &= \frac{1}{u_{2n}(t)} \quad n = 1, 2, \ldots \qquad u_1(t) = \beta - t + t_0 \end{aligned} \tag{6.6}$$

with $\beta = u_1(t_0) > 2$ an arbitrary parameter. According to the previous considerations we conclude that the weight function for the polynomials $P_n(x;t)$ with the recurrence coefficients (6.6) differs from the weight function (6.3) by the presence of the mass $M(t) = (t-t_0)/(\beta - t + t_0)$ inserted at $x = 0$.

From equations (6.6) we find the asymptotic behaviour of the recurrence coefficients for large n:

$$\begin{aligned} u_{2n} &= 1 + \frac{1}{n} + \frac{t-1}{tn^2} + \mathrm{O}\left(\frac{1}{n^3}\right) \\ u_{2n+1} &= 1 - \frac{1}{n} + \frac{1}{tn^2} + \mathrm{O}\left(\frac{1}{n^3}\right) \end{aligned} \tag{6.7}$$

where we have put $t_0 = 0$ for simplicity. We see from (6.7) that the first two terms of the expansions do not contain the mass M (or, equivalently, the time parameter t) in accordance with the general result (5.3). This dependence appears only in the terms $\sim 1/n^2$.

Note that the cases $\beta = 1$ and $\beta = 2$ correspond to the Chebyshev polynomials $U_n(x)$ and $T_n(x)$, respectively. In these special cases equations (6.6) (corresponding to the insertion of a discrete mass at the centre of the spectral interval) were obtained previously in [5, 9].

Another example is obtained if one takes the Hermite polynomials

$$H_{n+1} + nH_{n-1} = xH_n. \tag{6.8}$$

In this case

$$h_n = n! \qquad H_{2n}(0) = (-1)^n(2n-1)!! \qquad H'_{2n+1}(0) = (-1)^n(2n+1)!! \tag{6.9}$$

and using (4.18), (4.19) we obtain

$$\begin{aligned} u_{2n(t)} &= \frac{(t-1)(2n)!! - t(2n+1)!!}{(t-1)(2n-2)!! - t(2n-1)!!} \\ u_{2n+1}(t) &= \frac{2n(2n+1)}{u_{2n}(t)} \end{aligned} \tag{6.10}$$

where we have put $t_0 = 0$. Using Stirling's formula we find the following asymptotic behaviour for large n:

$$\begin{aligned} u_{2n}(t) &= 2n\left(1 + \frac{1}{2n} + \frac{\sqrt{\pi}(t-1)}{4tn^{3/2}} + \mathrm{O}\left(\frac{1}{n^2}\right)\right) \\ u_{2n+1}(t) &= (2n+1)\left(1 - \frac{1}{2n} - \frac{\sqrt{\pi}(t-1)}{4tn^{3/2}} + \mathrm{O}\left(\frac{1}{n^2}\right)\right). \end{aligned} \tag{6.11}$$

Again, as in the case of finite orthogonality interval, the value of the discrete mass only appears in the third terms of the approximation.

7. Connection with the Darboux transformation

In this section we show that the Uvarov–Chihara problem (as well as the solution of our integrable chain) is connected with the so-called Darboux transformations for orthogonal polynomials. Recall that the Darboux transformation (DT) for the discrete Schrödinger equation (DSE)

$$\psi_{n+1} + u_n\psi_{n-1} = x\psi_n \tag{7.1}$$

is defined [10–13] by

$$\tilde{\psi}_n = (\psi_n + A_n\psi_{n-2})F(x) \tag{7.2}$$

where $F(x)$ is an arbitrary function of the spectral parameter x and where the coefficients A_n obey the nonlinear recurrence relation

$$A_{n-1}(A_{n+1} - u_n) = A_n(A_{n-1} - u_{n-2}). \tag{7.3}$$

The function $\widetilde{\psi}_n$ obeys the DSE (7.1) with the same eigenvalue x, but with another potential, namely

$$\widetilde{u}_n = u_{n-2}\frac{A_n}{A_{n-1}}. \tag{7.4}$$

So far, the eigenfunctions ψ_n are arbitrary; interesting possibilities arise if one demands that both $\widetilde{\psi}_n$ and ψ_n be orthogonal polynomials in x. In [11, 12] we considered two such possibilities:

1. $\psi_n = P_n(x) \qquad \widetilde{\psi}_n = \widetilde{P}_n(x) \qquad F(x) \equiv 1.$

(This transformation was first considered by Geronimus [14]);

2. $\psi_n = P_n(x) \qquad \widetilde{\psi}_n = \widetilde{P}_{n-2}(x) \qquad F(x) = \dfrac{1}{(x^2 - \mu^2)}$

where μ is a constant outside the spectral interval. This transformation is well known as the Christoffel transform of orthogonal polynomials [1]. It also corresponds to passing to kernel polynomials (for details see [2]).

In both cases 1 and 2, provided that both the left- and right-hand sides of (7.3) are non-zero, equation (7.3) can be integrated once:

$$(A_{n+1} - u_n)(A_n - u_{n-1}) = -\mu^2 A_n \tag{7.5}$$

with the constant μ playing the role of the integral of (7.3), In these cases we can linearize equation (7.5) by using the substitution

$$A_n = -\frac{\varphi_n}{\varphi_{n-2}} \tag{7.6}$$

where φ_n is any solution of the DSE with μ as auxiliary spectral parameter

$$\varphi_{n+1} + u_n\varphi_{n-1} = \mu\varphi_n. \tag{7.7}$$

More precisely, in case 2 $\varphi_n = P_n(\mu)$.

Consider transformation of the weight functions under the Geronimus and Christoffel transformations. Let $w(x)$ be the weight function for the polynomials $P_n(x)$. Then under the Geronimus transformation [14] we have (up to an inessential common factor)

$$\widetilde{w}(x) = \frac{w(x)}{(x^2 - \mu^2)} + \frac{J}{2}\big(\delta(x-\mu) + \delta(x+\mu)\big) \tag{7.8}$$

where J is some constant specifying the masses added at the points $\pm\mu$; the arbitrariness of J is connected to the arbitrariness in the choice of the solution φ_n. In the case of Christoffel transformation[2] we have

$$\tilde{w}(x) = w(x)(x^2 - \mu^2). \tag{7.9}$$

It turns out that transformations 1 and 2 are reciprocal, as can easily be seen from equations (7.8) and (7.9).

There is yet another possibility for obtaining new orthogonal polynomials through the DT (7.2). Indeed, consider the choice

3. $\psi_n = P_n(x) \qquad \widetilde{\psi}_n = P_{n-1}(x) \qquad F(x) = \dfrac{1}{x}$

or, explicitly

$$\widetilde{P}_n(x) = \frac{(P_{n+1} + A_{n+1}P_{n-1})}{x}. \tag{7.10}$$

Since both $P_n(x)$ and $\widetilde{P}_n(x)$ must be symmetric orthogonal polynomials, we have

$$A_{2n+2} = -\frac{P_{2n+2}(0)}{P_{2n}(0)} = u_{2n+1}. \tag{7.11}$$

Hence from equations (7.11) and (7.10) we find

$$\widetilde{P}_{2n+1}(x) = P_{2n+1}(x). \tag{7.12}$$

Analogously, setting $n = 2k$ in (7.10) and using (7.12) we arrive at the relation

$$\widetilde{P}_{2n}(x) = \frac{P_{2n+1}(x) + \tilde{u}_{2n}P_{2n-1}(x)}{x}. \tag{7.13}$$

We can then identify equations (7.12) and (7.13) with equations (4.5) and (4.7) to obtain a solution of the Uvarov–Chihara problem. The Uvarov–Chihara problem is thus equivalent to this third type of Darboux transformation for symmetric orthogonal polynomials.

Finally, note that there is one more type of DT with $\mu = 0$:

$$\widetilde{P}_n = \frac{(P_{n+2} + A_n P_n)}{x^2}. \tag{7.14}$$

It can easily be seen from (7.14) (when $x = 0$) that the coefficients can be found explicitly:

$$A_{2n} = u_{2n+1} \qquad A_{2n-1} = -\frac{P'_{2n+1}(0)}{P'_{2n-1}(0)}. \tag{7.15}$$

The transformation (7.14) corresponds to the following transformation of the weight function:

$$\widetilde{w}(x) = x^2 w(x). \tag{7.16}$$

Acknowledgments

The authors are greatly indebted to the referee for numerous remarks and suggestions leading to improvements in the text of the paper.

The work of LV is supported in part through funds provided by NSERC (Canada) and FCAR (Quebec). The work of AZ was supported in part through funds provided by SCST (Ukraine) Project #2.4/197, INTAS-96-0700 grant and project 97-01-00281 supported by RFBR (Russia). AZ is grateful to the Centre de Recherches Mathématiques, Université de Montréal, for their hospitality.

References

[1] Szegö G 1975 *Orthogonal Polynomials (American Mathematical Society Colloquium Publications 23)* (Providence, RI: American Mathematical Society) 4th edn

[2] Chihara T S 1978 *An Introduction to Orthogonal Polynomials* (New York: Gordon and Breach)

[3] Kharchev S, Marshakov A, Mironov A, Orlov A and Zabrodin A 1991 Matrix models among integrable theories: forced hierarchies and operator formalism *Nucl. Phys.* B **366** 569–601

[4] Uvarov V B 1969 The connection between systems of polynomials orthogonal with respect to different distribution functions *USSR Comput. Math. Math. Phys.* **9** 25–36

[5] Chihara T S 1985 Orthogonal polynomials and measures with end point masses *Rocky Mountain J. Math.* **15** 705–19

[6] Baz A I', Zel'dovich Ya B and Perelomov A M 1971 *Scattering, Reactions and Decays in Non-relativistic Quantum Mechanics* (Moscow: Nauka) (in Russian)

[7] Geronimo J S and Nevai P G 1983 Necessary and sufficient conditions relating the coefficients in the recurrence formula to the spectral function for orthogonal polynomials *SIAM J. Math. Anal.* **14** 622–37

[8] Geronimus Ya L 1957 On some difference equations and corresponding systems of orthogonal polynomials *Zap. Mat. Otdel. Fiz.-Mat. Fak. Khar'kov Univ.* **25** 86–100

[9] Charris J A and Ismail M E H Sieved orthogonal polynomials. VII: Generalized polynomials mappings *Trans. Am. Math. Soc.* **340** 71–93

[10] Matveev V B and Sall' M A 1991 *Darboux Transformations and Solitons (Springer Series in Nonlinear Dynamics)* (Berlin: Springer)

[11] Spiridonov V and Zhedanov A 1995 Discrete Darboux transformation, discrete-time Toda lattice, and the Askey–Wilson polynomials *Methods Appl. Anal.* **2** 369–98

[12] Spiridonov V, Vinet L and Zhedanov A 1996 Discrete Schrödinger equation, Darboux transformation and orthogonal polynomials *CRM Proc. Lect. Notes* **9** 363–9

[13] Spiridonov V, Vinet L and Zhedanov A 1997 Spectral transformations, self-similar reductions and orthogonal polynomials *J. Phys. A: Math. Gen.* **30** 7621–37

[14] Geronimus Ya L 1940 On the polynomials orthogonal with regard to a given sequence of numbers *Zapiski NII Mat. i Mehan. Khar'kov Univ.* ser. 4 **17** 3–18